Statement:
Let \( n > 1 \) be an integer, and let \( a \) be a fixed integer. Prove or disprove that the set
\[
H = \{ x \in \mathbb{Z} \mid ax \equiv 0 \pmod{n} \}
\]
is a subgroup of \( \mathbb{Z} \) under addition.

Proof:
Let \( x = 0 \). Then \( a(0) \equiv 0 \pmod{n} \), so \( 0 \in H \). Therefore \( H \neq \emptyset \). Now suppose \( x, y \in H \). This implies that \( ax \equiv 0 \pmod{n} \) and \( ay \equiv 0 \pmod{n} \). But then \( (ax + ay) \equiv 0 \pmod{n} \) which implies that \( a(x + y) \equiv 0 \pmod{n} \). Since \( (x + y) \in \mathbb{Z} \) and \( a(x + y) \equiv 0 \pmod{n} \), so \( (x + y) \in H \). Therefore, \( H \) is a subgroup of \( \mathbb{Z} \).

p160n10 Statement and Review
E.H. Moore

Statement:
Let \( n > 1 \) be an integer, and let \( a \) be a fixed integer. Prove or disprove that the set
\[
H = \{ x \in \mathbb{Z} \mid ax \equiv 0 \pmod{n} \}
\]
is a subgroup of \( \mathbb{Z} \) under addition.

Review:
The first three sentences of Cayley’s proof correctly show that the first condition of the definition of a subgroup- that \( H \) is nonempty- is satisfied.

The next part of the proof also correctly shows that the group is closed under the operation of addition, which satisfies the second condition of the definition. The only thing that could be slightly changed with this part is the second to last sentence. It could say “Since \( (x + y) \in \mathbb{Z} \) and \( a(x + y) \equiv 0 \pmod{n} \), then \( (x + y) \in H \) and therefore \( H \) is closed under the operation of addition.”

Cayley ends his proof here, but he is still missing one of the conditions needed to show that \( H \) is a subgroup, which is that \( x \in H \) implies \( x^{-1} \in H \).
Professor’s Comments:

Moore is quite correct that Cayley has omitted the proof that \( H \) is closed under inversion. In that proof, however, we should denote the inverse of an element \( x \) by \( -x \) rather than by \( x^{-1} \), since the operation is the familiar operation of addition. Moore also offers a better way to word the conclusion of closure of \( H \) under addition, and this should be incorporated into the final solutions.

Also, Cayley begins by defining what he means by “\( x \),” but then does not use \( x \) in the next two sentences. It would be better to rewrite the second sentence, say, as “Then \( ax = a(0) \equiv 0 \pmod{n} \), so \( 0 \in H \).”

A few modifications will turn Cayley’s solution into one that is perfectly correct.

Grades:

Cayley: 8/10 (minus 40% for lateness) = 4.8/10
Moore: 5/5
Gauss: 0/5 (review not submitted)