So far this year (and probably for most of the time you’ve been doing mathematics) you’ve considered points and graphs in the **Cartesian coordinate system** also known as the ________________ coordinate system. For the most part, these coordinates work very well. But not all the time. So for the next several class periods, we’re going to discuss another system: ___________ coordinates.

**The Basics**

In the Cartesian coordinate system, we have a special point called the origin, and two axes (vertical and horizontal) that we use to locate points in the plane.

In polar coordinates, we have a special point (called the ___________ ) and one horizontal ray (called the ______________________ ); this ray is horizontal with the endpoint at the pole, and extending infinitely to the right. The coordinates of a point have the form _____________ , where \(|r|\) is the _______________________ , and \(\theta\) is the angle through which the ______________ must be rotated ______________________ to reach the point.

**Examples:**

A: \((4, 0)\)  
B: \((3, \frac{\pi}{3})\)  
C: \((4, \frac{3\pi}{4})\)  
D: \((5, \frac{7\pi}{6})\)  
E: \((1, \frac{3\pi}{2})\)  
\(= (1, -\frac{\pi}{2})\)  

Polar coordinates are not unique. 

E: \((-1, \frac{\pi}{2})\)  
\(\equiv -1 \text{ on polar axis; rotate } \frac{\pi}{2}\)
Plot these points:

\[ A = \left(3, -\frac{\pi}{6}\right)_p \]
\[ B = (1, -3\pi)_p \]
\[ C = \left(0, \frac{3\pi}{4}\right)_p \]
\[ D = \left(-5, \frac{13\pi}{4}\right)_p \]

\[ \frac{13\pi}{4} = \frac{12\pi}{4} + \frac{\pi}{4} = \frac{3\pi}{4} + \frac{\pi}{4} \]

Converting Between Cartesian and Polar Coordinates

Consider a point \( P \) with Cartesian coordinates \((x, y)\) and polar coordinates \((r, \theta)\).

* \( x = r \cos \theta \) \quad (\cos \theta = \frac{x}{r})
* \( x^2 + y^2 = r^2 \) \implies \( r = \pm \sqrt{x^2 + y^2} \)
* \( y = r \sin \theta \) \quad (\sin \theta = \frac{y}{r})

If we know \((r, \theta)_P\), use:

\[ x = r \cos \theta, \quad y = r \sin \theta \]

**Example:** Find the Cartesian coordinates of the following points.

\[ \left(2, -\frac{\pi}{6}\right)_P \]

\[ x = 2 \cos \left(-\frac{\pi}{6}\right) = 2 \left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{3} \quad x = r \cos \theta, \quad y = r \sin \theta \]

\[ y = 2 \sin \left(-\frac{\pi}{6}\right) = 2 \left(-\frac{1}{2}\right) = -1 \]

So \( (2, -\frac{\pi}{6})_P = (-\sqrt{3}, -1)_R \)

\[ (-1, 6\pi)_P \]

\[ x = -1 \cos (6\pi) = -1(1) = -1 \]
\[ y = -1 \sin (6\pi) = -1(0) = 0 \]

\[ (-1, 6\pi)_P = (-1, 0)_R \]
Example: Find polar coordinates for each of the following points.

\[(6, 6\sqrt{3}) \quad \text{Quad I} \quad r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x}
\]

\[\tan \theta = \frac{y}{x} = \frac{6\sqrt{3}}{6} = \sqrt{3} \quad \text{in } [0, 2\pi], \theta = \frac{\pi}{3}, \frac{4\pi}{3}
\]

\[r^2 = (6)^2 + (6\sqrt{3})^2 = 144 \quad r = \pm 12\]

In Quadrant I: \((12, \frac{\pi}{3})_p \quad \text{or} \quad (-12, \frac{4\pi}{3})_p\)

\[(-4, 4) \quad \text{Quadrant II} \quad \tan \theta = \frac{y}{-x} = -1 \quad \text{in } [0, 2\pi], \theta = \frac{3\pi}{4}, \frac{7\pi}{4}
\]

\[r^2 = (-4)^2 + 4^2 = 32 \quad r = \pm \sqrt{32} = \pm 4\sqrt{2}\]

In Quadrant II: \((\sqrt{32}, \frac{3\pi}{4})_p \quad (-\sqrt{32}, \frac{7\pi}{4})_p\)

Graphing Polar Functions

In “polar” mode, your calculator can graph equations of the form \(r = f(\theta)\). It’s really, really important that you use a Square viewing window, so that circles look like circles.
Examples: Graph the following equations.

A. \( r = 2 \cos \theta \)  
B. \( r = -3 \sin \theta \)  
C. \( r = 2 \)  
D. \( r = \cos (4\theta) \)  
E. \( r = 2 \sin (3\theta) \)  
F. \( r = 1 + \cos x \)  
G. \( r = \frac{1}{2} + \sin x \)  
H. \( r = 1 - 2 \sin x \)  
I. \( r^2 = \cos 2\theta \)  
J. \( r = \ln \theta, \theta \geq 1 \)  
K. \( r = \sec \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2} \)  
L. \( \theta = \frac{\pi}{6} \)

\[ r = \sqrt{\cos 2\theta} \]

Cardiod heart  
“rose”  
Lemniscate “ribbon”  
“Snail”  
Spiral

\[ r = 3 - 3 \sin \theta + \frac{\sin \theta \sqrt{\cos \theta}}{\sin \theta + 1.3} \]
Example: Write an equation in Cartesian coordinates that describes the curves given below in polar coordinates.

\[ r \cos \theta = 8 \]

\[ r = 3 \sin \theta \]

\[ r^2 = \cos 2\theta \]

\[ \theta = -\frac{\pi}{4} \]

Example: Write an equation in polar coordinates that describes the curves given below in Cartesian coordinates.

\[ x^2 + y^2 = 16 \]

\[ y = 3x^2 \]
Parametric Representation of Polar Curves

Any curve given by \( r = f(\theta) \) can be parametrized in the Cartesian plane by

\[
\begin{align*}
  x &= \quad \\
  y &= 
\end{align*}
\]

Slope

This brings us to the idea of slope which is a uniquely Cartesian concept. Still, if we have the graph of a polar equation, and the graph is smooth, we can draw tangent lines, and discuss the slope of those lines.

From our look at parametric equations, we know that if \( r = f(\theta) \), then

\[
\begin{align*}
  x &= \quad \\
  y &= 
\end{align*}
\]

So \( \frac{dy}{dx} = \)

Example: Find the slope of the line that is tangent to the graph of \( r = 2 \sin(3\theta) \), when \( \theta = \frac{\pi}{3} \).