FINITE LOCALIZATIONS AND THE CLASSICAL ADAMS SPECTRAL SEQUENCE

PAUL SHICK

January 22, 2001

The goal of this note is to present some of the relationship between some "old-fashioned" constructions in homological algebra and the recent finite localization functors of Mahowald-Sadofsky, Ravenel, Miller, et al ([7], [14], [9].) The main result verifies a conjecture of Mahowald and Sadofsky (for finite spectra,) yielding a somewhat geometric interpretation of the so-called "v_n-periodic Ext-groups." It is a pleasure to thank Mark Hovey, Mark Mahowald, John Palmieri and Hal Sadofsky for helpful conversations and correspondence.

The homological algebra referred to above is concerned with

$$\mathrm{Ext}_A^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p),$$

the E_2 term of the classical Adams spectral sequence for $\pi_*(S^0)$ for any prime p, where A is the mod p Steenrod algebra. One way to study this is by using appropriate finite subalgebras. For each $n \geq 0$, let $A(n)$ denote the finite subHopf algebra of the Steenrod algebra generated by

$$Sq^0, Sq^1, \ldots, Sq^{2^n}$$

if $p = 2$ and by

$$\beta, P^1, \ldots, P^{p^{n-1}},$$

if p odd. Then the Hopf algebra cohomology

$$H^*(A(n)) = \mathrm{Ext}_{A(n)}(\mathbb{Z}/p, \mathbb{Z}/p)$$

is of interest for several reasons:

- Since $\varprojlim_n \mathrm{Ext}_{A(n)}(\mathbb{Z}/p, \mathbb{Z}/p) = \mathrm{Ext}_A(\mathbb{Z}/p, \mathbb{Z}/p)$, we get complete information about our goal.
- For each $n \geq 0$, $\mathrm{Ext}_{A(n)}$ is effectively computable by "finite" methods.

- These Ext-groups can be used to study the "chromatic" approach to homotopy in the setting of the classical Adams spectral sequence. In particular, $\mathrm{Ext}_{A(n)}$ is closely related to the "periodic" spectra $BP\langle n\rangle$, as shown below.

For each prime p, the Brown-Peterson spectrum BP is well-known, with $H^*(BP) = A//E(Q_0, Q_1, \dots)$, where Q_n is the n^{th} Milnor generator of the mod p Steenrod algebra. The associated Johnson-Wilson spectrum is $BP\langle n\rangle$,, with $H^*(BP\langle n\rangle) = A//E_n$ where we use the notation $E_n = E(Q_0, Q_1, \dots, Q_n)$, and $H^*(-)$ denotes cohomology with $\mathbb{Z}/p$ coefficients. So we can calculate $\pi_*(BP\langle n\rangle)$ from the classical Adams spectral sequence:

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}_A^{s,t}(H^*(BP\langle n\rangle), \mathbb{Z}/p) \\ &= \mathrm{Ext}_A^{s,t}(A//E_n, \mathbb{Z}/p) \\ &\cong \mathrm{Ext}_{E_n}(\mathbb{Z}/p, \mathbb{Z}/p) \\ &\cong \mathbb{Z}/p[q_0, q_1, \dots, q_n], \end{aligned}$$

where the class $q_i \in \mathrm{Ext}^{1,2p^i-1}$. Since these generators are concentrated in even degrees, we conclude that $E_2 = E_\infty$ here. Further, because the E_2 generator q_0 corresponds to the Bockstein operation, it represents the element $p \in \pi_0(BP\langle n\rangle)$. So we see easily that

$$\pi_*(BP\langle n\rangle) \cong \mathbb{Z}_{(p)}[v_1, \dots, v_n],$$

where $|v_i| = 2p^i - 2$. The periodic version of the Johnson-Wilson spectrum is $E(n)$, with

$$E(n)_* = \mathbb{Z}_{(p)}[v_0, v_1, \dots, v_n, v_n^{-1}].$$

The inclusion $i : E_n \hookrightarrow A(n)$ induces a restriction homomorphism in cohomology

$$i^* : \mathrm{Ext}_{A(n)}(\mathbb{Z}/p, \mathbb{Z}/p) \to \mathrm{Ext}_{E_n}(\mathbb{Z}/p, \mathbb{Z}/p).$$

We use this to see clearly the link between $BP\langle n\rangle$ and $\mathrm{Ext}_{A(n)}$. The main result of [15] is the following.

Theorem 1. *For p odd, there exists a polynomial subalgebra*

$$\mathbb{Z}/p[v_0, v_1^{p^n}, v_2^{p^{n-1}}, \dots, v_n^p] \subset Ext_{A(n)}(\mathbb{Z}/p, \mathbb{Z}/p),$$

where the generators restrict to the obvious classes in $H^*(E_n)$.

The proof uses a careful analysis of the Cartan-Eilenberg spectral sequence for the extension

$$\mathbb{F}_p \to P_n \to A(n)_* \to E^n \to \mathbb{F}_p$$

where P_n is the truncated polynomial algebra on the ξ_is and E^n is the exterior algebra on the τ_is.

For the prime 2, the situation is more difficult. The following is proved in [8].

Theorem 2. *There exists a polynomial subalgebra*

$$\mathbb{Z}/p[v_0, v_1^{N_1}, v_2^{N_2}, \dots, v_n^{N_n}] \subset Ext_{A(n)}(\mathbb{Z}/2, \mathbb{Z}/2),$$

where the generators restrict to the obvious classes in $H^*(E_n)$.

The proof uses results of Lin [3] and Wilkerson [16] that show that the restriction homomorphism i^* is onto in infinitely many positive degrees. Note that these generators are only defined up to cosets for $p = 2$. Using a spectral sequence based on a Koszul-type resolution of $A(n)//A(n-1)$, we showed that the exponent N_n could be identified:

$$v_n^{2^{n+1}} \in \mathrm{Ext}_{A(n)}(\mathbb{Z}/2, \mathbb{Z}/2).$$

The other exponents were more mysterious. In answer to a recent question of W. Singer, we have the following.

Theorem 3. *For all $i \geq n+1$, there exists r_i such that*

$$v_i^{2^{r_i}k} \in Ext_{A(n)}(\mathbb{Z}/2, \mathbb{Z}/2),$$

for all $k \geq 1$.

Proof: Consider the May-Ivanovskii spectral sequence for $H^*(A(n))$. This collapses at a finite E_r, by the results of Bajer and Sadofsky ([1]). The element v_i corresponds to $h_{i,0}$ in the E_1 term. Because the differentials are derivations, we have room for only finitely many:

$$\begin{array}{rcl} v_i & \mapsto & a_1 \\ v_i^2 & \mapsto & a_2 \\ & \vdots & \\ v_i^{2^{r_i}} & \mapsto & 0. \end{array}$$

Thus the appropriate powers of the elements are cycles in the spectral sequence. Mapping to the same spectral sequence for $H^*(E_n)$ shows that these elements are never boundaries. This completes the proof of the theorem.

We use these polynomial subalgebras in $H^*(A(n))$ to define "v_n-periodic" Ext, $v_n^{-1}\mathrm{Ext}_A(\mathbb{Z}/p, \mathbb{Z}/p)$, as follows:

$$\begin{tikzcd}
\mathrm{Ext}_A(\mathbb{Z}/p,\mathbb{Z}/p) \arrow[d] & \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathrm{Ext}_{A(k+1)}(\mathbb{Z}/p,\mathbb{Z}/p) \arrow[r] \arrow[d] & v_n^{-1}\mathrm{Ext}_{A(k+1)}(\mathbb{Z}/p,\mathbb{Z}/p) \arrow[d] \\
\mathrm{Ext}_{A(k)}(\mathbb{Z}/p,\mathbb{Z}/p) \arrow[r] \arrow[d] & v_n^{-1}\mathrm{Ext}_{A(k)}(\mathbb{Z}/p,\mathbb{Z}/p) \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathrm{Ext}_{A(n)}(\mathbb{Z}/p,\mathbb{Z}/p) \arrow[r] & v_n^{-1}\mathrm{Ext}_{A(n)}(\mathbb{Z}/p,\mathbb{Z}/p).
\end{tikzcd}$$

So we define

$$v_n^{-1}\mathrm{Ext}_A(\mathbb{Z}/p,\mathbb{Z}/p) := \varprojlim_k \{v_n^{-1}\mathrm{Ext}_{A(k)}(\mathbb{Z}/p,\mathbb{Z}/p)\}.$$

Taking the inverse limit, we obtain a "localization" map

$$f_n : \mathrm{Ext}_A(\mathbb{Z}/p,\mathbb{Z}/p) \longrightarrow v_n^{-1}\mathrm{Ext}_A(\mathbb{Z}/p,\mathbb{Z}/p),$$

defining v_n-periodic and v_n-torsion in $\mathrm{Ext}_A(\mathbb{Z}/p,\mathbb{Z}/p)$ in terms of the kernel of f_n.

The following is proved in [15] and [8].

Theorem 4. *If a class $a \,\epsilon\, Ext_A(\mathbb{Z}/p,\mathbb{Z}/p)$ is v_n-periodic, then a is also v_{n+k}-periodic for all $k \geq 0$. Equivalently, if a is v_n-torsion, then a is also v_i-torsion for all $i \leq n$.*

Note that this sets up a "chromatic filtration" of $\mathrm{Ext}_A\mathbb{Z}/p$, closely related to that subsequently defined by Palmieri ([10]).

We wish to investigate the relationship between this "v_n-periodic Ext" and the finite or telescopic localizations that have been the subject of much study in recent years. Such functors have been considered at least partly as a consequence of calculations that show that the Telescope Conjecture ([13]) is most likely false ([14],[6],) providing a construction which agrees with the geometrically defined telescope, but which preserves the desirable functorial properties of the usual Bousfield localization.

One version of this sort of functor is due to Mahowald and Sadofsky, who define the "telescopic localization" of a spectrum. To explain their construction, we need to recall some results that follow from the Nilpotence Theorem, most of which are due to Hopkins and Smith ([4].) We recall that a finite spectrum X is said to be of type n if $K(n-1)_*(X) = 0$ and $K(n)_*(X) \neq 0$, where $K(j)$ is the j^{th} mod p Morava K-theory, with $K(j)_* = \mathbb{F}_p[v_j, v_j^{-1}]$. Such a type n finite complex always has a "v_n-self map:

$$v : \Sigma^i X \to X,$$

which induces an isomorphism in $K(n)_*(X)$. For X of type n, the v_n-telescope of X is

$$\mathrm{Tel}_{v_n}(X) = v_n^{-1}X = \varinjlim(X \to \Sigma^{-i}X \to \Sigma^{-2i}X \to \cdots)$$

for any v_n self map $v : \Sigma^i X \to X$. All such v_n-telescopes of X are easily seen to be homotopy equivalent, so that $\mathrm{Tel}_{v_n}(X)$ is well-defined.

We further recall some basics of Bousfield's theory. A spectrum X is said to be E-local if

$$[W, X]_* = 0 \text{ whenever } E_*W = 0.$$

The Bousfield localization of X with respect to E is a map

$$L : X \to L_E X,$$

where L_E is E-local and E_*L is an isomorphism. Two spectra X and Y are Bousfield equivalent if

$$X \wedge W \simeq * \Longleftrightarrow Y \wedge W \simeq *.$$

Two spectra are said to be in the same Bousfield class if they are Bousfield equivalent. Bousfield localization with respect to the periodic Johnson-Wilson spectrum $E(n)$ has been so important to the "chromatic" approach to topology that the notation

$$L_n X := L_{E(n)} X$$

has become standard. See [13] for the origins of this school. Note that $L_n X = L_{(K(0)\vee\cdots\vee K(n))} X$ for all $n \geq 0$.

Mahowald and Sadofsky base their definition of telescopic localization on the following observation: for all $n \geq 0$, for any finite complex X of type n, the Bousfield class of $\mathrm{Tel}_{v_n}(X)$ depends only on n. We hereafter denote any such telescope by T_n. For any spectrum X (finite or not,) we define the n^{th} telescopic localization of X as

$$L_n^f X := L_{(T_0 \vee T_1 \vee \cdots \vee T_n)} X.$$

One can show easily that if X is finite type n, then

$$\mathrm{Tel}_{v_n}(X) \simeq L_{T_n} X \simeq L_n^f X.$$

The definition of $L_n^f X$ above can be shown to agree with the finite localization functors L_n^f, given by Ravenel ([14]) and Miller ([9].) Ravenel's definition involves constructing an appropriate spectrum $L_n^f S^0$ as a limit of certain finite complexes, then defining

$$L_n^f X = L_{L_n^f S^0} X,$$

for all spectra X. Miller's definition follows the original ideas behind usual Bousfield localization with respect to the periodic Johnson-Wilson spectrum $E(n)$, considering the cofiber of

$$\bigvee_{(W\ finite,\, E(n)_* W = 0)} W \to X.$$

Further, one can show that L_n^f is smashing. That is, $L_n^f X = (L_n^f S^0) \wedge X$, for all spectra X.

The Telescope Conjecture ([13]) predicts that for any finite complex of type n, the localization map

$$Tel_{v_n} X \xrightarrow{L} L_n X$$

should be an equivalence. At the time the conjecture was proposed, it was thought that only the localization functors L_n were smashing. Since finite localization with respect to $E(n)$ agrees with the v_n-telescope, some have reformulated the Telescope Conjecture as "In the stable homotopy category, all smashing localizations are finite." (See [5].)

Mahowald and Sadofsky found a more geometric construction of $L_n^f X$. For an ordered index $I = (i_0, i_1, \ldots, i_n)$, let

$$M(I) = M(p^{i_0}, v_1^{i_1}, \ldots, v_n^{i_n})$$

denote any choice of finite spectrum with

$$BP_*(M(p^{i_0}, v_1^{i_1}, \ldots, v_n^{i_n})) = BP_*/(p^{i_0}, v_1^{i_1}, \ldots, v_n^{i_n}),$$

a so-called "generalized Moore space." With the usual order on these indices, we have maps

$$f_J^I : M(I) \to \Sigma^{|I-J|} M(J)$$

commuting with the projection onto the top cell, whenever $J \leq I$, where

$$|I| = i_1 2(p-1) + i_2 2(p^2-1) + \cdots + i_n 2(p^n-1).$$

Let $\overline{M}(I)$ denote the fiber of the projection onto the top cell

$$M(I) \xrightarrow{p} S^{|I|+n+1},$$

So we have a cofiber sequence

$$S^{n+|I|} \xrightarrow{g_I} \overline{M}(I) \to M(I),$$

for all I.

Theorem 5. *(Mahowald-Sadofsky, [7]). The map*

$$S^0 \xrightarrow{g_I} \varinjlim_I \Sigma^{-n-|I|} \overline{M}(I)$$

is L_n^f localization.

This is a vast generalization of earlier work of Davis, Mahowald and Miller on limits of stunted projective spaces ([2].) Since L_n^f is smashing, it gives a construction of $L_n^f X$ for all X. However, this looks inherently incalculable.

The surprising part of the Mahowald-Sadofsky program is their construction of an Adams spectral sequence converging to $\pi_*(L_n^f X)$. The construction is quite technical, requiring a modified Adams tower and a lot of work with "stably finite" A-modules and spectra (ala Palmieri-Sadofsky [12].) In the end, they identify the E_2 term of the spectral sequence in the following manner.

First, they work in category of comodules over the coalgebra A_*. Let St denote the category of chain complexes of A_*-comodules, with $G_\bullet$ an injective resolution of $\mathbb{Z}/p = H_*(S^0)$. For any spectrum X,

$$E_2^{clASS}(X) = \mathrm{Ext}_{A_*}(\mathbb{Z}/p, H_*(X)) \cong [G_\bullet, N_\bullet],$$

where $N_\bullet$ = injective resolution of $H_*(X)$ and $[-,-]$ denotes chain homotopy classes of maps in St. This category is explored in great detail in [11].

The "finite localization" Adams spectral sequence uses a modified Adams resolution for the finite spectrum $\overline{M}(I)$ (for appropriate I) and then takes a direct limit, resulting in a spectral sequence converging to $\pi_*(L_n^f X)$. Mahowald and Sadofsky identify the resulting E_2 term as

$$E_2^{L_n^f ASS}(X) \cong [G_\bullet, L_J(N_\bullet)],$$

where L_J denotes a certain chain complex localization that looks like an algebraic analog of the "finite localization" of Miller, et al. In particular, one can choose $J_\bullet =$ the finite chain complex given by a modified Adams resolution for $M(U^I)$, for some particular I, a finite complex of type $n+1$.

The connection between the ideas of Mahowald and Sadofsky, among others, and the homological algebra presented earlier is the following conjecture.

Conjecture 6. *(L_n^f-Ext Conjecture, Mahowald-Sadofsky* [7]*) For X any spectrum (in the usual stable homotopy category,)*

$$E_2^{L_n^f ASS}(X) \cong v_n^{-1} Ext_A(H^*X, \mathbb{Z}/p),$$

where

$$v_n^{-1} Ext_A(\mathbb{Z}/p, \mathbb{Z}/p) := \varprojlim_k \{v_n^{-1} Ext_{A(k)}(\mathbb{Z}/p, \mathbb{Z}/p)\},$$

as above.

Mahowald and Sadofsky cite some evidence for the conjecture. They note that it's true for $n = 0, 1$ by direct computation for $X = S^0$ (then use the fact that L_n^f is smashing to extend to all spectra.) Further, they prove that the conjecture holds for X a finite spectrum of type n, with the added condition of H^*X having appropriate P_s^t-homology groups vanish. However, the conjectured isomorphism is between an object defined as a direct limit (the $E_2^{L_n^f ASS}$) and an inverse limit (the v_n-periodic Ext-group,) so one is uninclined to be optimistic about the conjecture.

Here the finite localization L_n^f is a spectrum level construction, (i. e. in the usual stable homotopy category,) with analogues in several appropriate algebraic settings. Note that we have a number of options on how to view the functor L_n^f, from the work of Miller, Ravenel, Mahowald and Sadofsky:

- As a direct limit (colimit) of cofibers built from type $n+1$ finites.
- As a Bousfield localization w.r.t. $L_n^f S^0$, which can be constructed as a direct limit of $\overline{M}(I)$s.
- As a **finite** localization away from a type $n+1$ finite.
- As a **finite** localization with respect to $E(n)$ or $K(\leq n)$, where $E(n)$ is the periodic Johnson-Wilson spectrum and $K(\leq n)$ is a shorthand notation for $K(0) \vee \cdots \vee K(n)$.
- As Bousfield localization with respect to $T(\leq n)$, a wedge of v_i-telescopes of finite type i complexes, with $i \leq n$.

Again, each spectrum level construction has an appropriate counterpart in appropriate algebraic settings.

The most convenient framework in which to explore these ideas is given in Palmieri's opus [11]. As above, let St denote the homotopy category whose objects are cochain complexes of injective graded comodules over the dual Steenrod algebra A_*, with morphisms cochain homotopy classes of graded maps of complexes. Let $St(i)$ denote the obvious analog of St over the dual of the finite Hopf algebra $A(i)$. In these categories, the sphere S (actually $S^{*,*}$, but we'll suppress the bigrading whenever possible) is an injective resolution of $\mathbb{Z}/p$, which we denote by $G(\infty)^\bullet \in St$, as above, or by $G(k)^\bullet \in St(k)$. For a cochain complex $X^\bullet \in St$, which is a resolution of a comodule M, the homotopy groups are just

$$\pi_{**}(X^\bullet) = [G(\infty)^\bullet, X^\bullet]_* = \mathrm{Ext}^{*,*}_{A_*}(\mathbb{Z}/p, M),$$

For a cochain complex $X^\bullet \in St(k)$,

$$\pi_{**}(X^\bullet) = [G(k)^\bullet, X^\bullet]_* = \mathrm{Ext}^{*,*}_{A(k)_*}(\mathbb{Z}/p, M).$$

We'll attempt as much as possible to be careful about which category we're working in, to avoid confusion about what $\pi_{**}(-)$ means. In particular, we adopt the following convention: for a cochain complex over A_*, $X^\bullet \in St$, we'll use $X^\bullet_k$ to denote $X^\bullet$ thought of as an $A(k)_*$-resolution. Further following Palmieri's notation, let $HA(i)^\bullet$ denote an injective resolution of $A_*\square_{A(i)_*}\mathbb{Z}/p$, so that $\pi_{**}(HA(i)^\bullet) = \mathrm{Ext}_{A(i)_*}(\mathbb{Z}/p, \mathbb{Z}/p)$. We also have $St(k)$ analogs of $HA(i)^\bullet$, for $k \geq i$, which are not just given by taking $A_*\square_{A(i)_*}\mathbb{Z}/p$, as $A(k)$-modules. Instead, in $St(i)$, $HA(i)^\bullet_i = G(i)^\bullet$, and in $St(k)$, $HA(i)^\bullet_k$ is an injective resolution of $A(k)_*\square_{A(i)_*}\mathbb{Z}/p$, so that the homotopy groups in $St(k)$ are again just $\mathrm{Ext}_{A(i)_*}(\mathbb{Z}/p, \mathbb{Z}/p)$.

There are obvious analogues of the L_n^f functor in both St and $St(k)$, for all k, which we'll denote by L_n^f and $L_{n,k}^f$, respectively. In particular, if X is a spectrum (in the usual stable category) and $C^\bullet$ is an A_*-resolution of H_*X, then

$$\pi_{**}(L_n^f C^\bullet) = E_2^{L_n^f ASS}(X).$$

With this form of Palmieri's notation, we can define $v_n^{-1}X^\bullet$, for any $X^\bullet \in St$, in several ways:

$$v_n^{-1}X^\bullet := \varprojlim_k (v_n^{-1}HA(k) \wedge X^\bullet_k)$$

and

$$v_n^{-1}X^\bullet := \varprojlim_k (v_n^{-1}HA(k)) \wedge X^\bullet = (v_n^{-1}S^0) \wedge X^\bullet.$$

For finite $X^\bullet$s, these two definitions agree, as shown in [11]. A third possibility is to remember that in $St(k)$, the sphere $S^0_k = HA(k)^\bullet = G(k)^\bullet$ has a v_n self map, so that for any spectrum $X^\bullet_k \in St(k)$, we can form the v_n-telescope $Tel_{v_n}(X^\bullet_k)$, using the self map

$$S^{|v_n^L|} \wedge X \xrightarrow{v_n^L} S^0 \wedge X = X.$$

So for any spectrum $X^\bullet \in St$, we have another possible definition:

$$v_n^{-1}X^\bullet := \varprojlim_k Tel_{v_n}(X^\bullet_k).$$

Lemma 7. *For a finite spectrum $X^\bullet \in St$, the three definitions of $v_n^{-1}X^\bullet$ agree.*

Proof: For finite $X^\bullet$s, the first two definitions agree, by Palmieri's work. For any finite $X^\bullet$, we have

$$\begin{aligned}
v_n^{-1}X^\bullet &:= \varprojlim_k v_n^{-1}(A\Box_{A(k)}X^\bullet_k) \\
&= \varprojlim_k A\Box_{A(k)}(\mathrm{Tel}_{v_n}X^\bullet_k) \\
&= \varprojlim_k HA(k)^\bullet \wedge (\mathrm{Tel}_{v_n}X^\bullet_k) \\
&\simeq \varprojlim_k (\mathrm{Tel}_{v_n}X^\bullet_k) \\
&= \varprojlim_k v_n^{-1}X^\bullet
\end{aligned}$$

since $\varprojlim_k HA(k)^\bullet = S^0 = G(\infty)^\bullet$ in St.

So we can think about $v_n^{-1}X^\bullet$ in several ways, all of which agree if X is a finite spectrum. Of course, we're really interested in

$$\begin{aligned}
v_n^{-1}\mathrm{Ext}_A(M, \mathbb{Z}/p) &:= \varprojlim_k v_n^{-1}\mathrm{Ext}_{A(k)}(M_k, \mathbb{Z}/p) \\
&= \varprojlim_k \pi_{**}(v_n^{-1}HA(k)^\bullet \wedge X^\bullet_k) \\
&= \pi_{**}((\varprojlim_k v_n^{-1}HA(k)) \wedge X^\bullet_k) \\
&= \pi_{**}(v_n^{-1}S^0 \wedge X^\bullet),
\end{aligned}$$

where $X^\bullet$ is an injective A_*-resolution of some finite comodule M. (We're restricting to finites here to make everything in sight agree.)

Mahowald and Sadofsky's conjecture would follow from the following:

Conjecture 8. *For any $X^\bullet \in St$,*

$$L_n^f X^\bullet \cong v_n^{-1} X^\bullet.$$

One might also conjecture that this isomorphism holds at the $St(k)$ level, asking if

$$L_{n,k}^f X_k^\bullet \cong v_n^{-1} X_k^\bullet???$$

for all $X_k^\bullet \in St(k)$. Looking at this question forces one to think about all sorts of interesting phenomena.

First, we have the following result, probably originally due to Mark Hovey (unpublished):

Lemma 9. *In $St(k)$, $v_n^{-1} X_k^\bullet$ is just finite localization of $X_k^\bullet$ away from the cofiber of the v_n element in the sphere, $HA(k)/v_n$.*

This is just "algebraic localization" of the unit in the category $St(k)$, so the proof is done in Hovey-Palmieri-Strickland ([5].) Here's a brief sketch. The category $St(k)$ is a unital algebraic stable homotopy category (in HPS terms,) with sphere $S = HA(k) = G(k)$. There's a power of v_n in $\pi_{**}(HA(k))$, which we'll denote by v_n to avoid unnecessary complications. We have the usual cofibration $HA(k) \xrightarrow{v_n} HA(k) \to HA(k)/v_n$. Recall that finite localization away from $HA(k)/v_n$ is a smashing finite localization with acyclics generated by $HA(k)/v_n$. Note that the functor $v_n^{-1}\pi_{**}(-)$ is a homology theory on $St(k)$, since it preserves coproducts and is exact as a functor of $\pi_{**}(HA(k))$-modules. So

$$v_n^{-1}\pi_{**}(X_k^\bullet) = v_n^{-1}\pi_{**}(HA(k) \wedge X_k^\bullet) = v_n^{-1}\pi_{**}(L_{(\text{away } HA(k)/v_n)} X_k^\bullet).$$

Further, $[HA(k)/v_n, L_{(\text{away } HA(k)/v_n)} X_k^\bullet] = 0$ for any $X_k^\bullet \in St(k)$, so that multiplication by v_n is an isomorphism on $\pi_{**}(L_{(\text{away } HA(k)/v_n)} X_k^\bullet)$. Hence

$$v_n^{-1}\pi_{**}(L_{(\text{away } HA(k)/v_n)} X_k^\bullet) = \pi_{**}(L_{(\text{away } HA(k)/v_n)} X_k^\bullet).$$

This completes the proof of the Lemma.

Recall that in $St(k)$ or St, $L_{n,k}^f$ (resp. L_n^f) is finite localization away from an injective resolution of $F(n+1)$, a finite complex of type n+1. This is easy to see, since the finitely-$K(\leq n)$-locals are exactly the finitely-$F(n+1)$-acyclics. (Recall that in the category St, we replace $K(\leq n)$ with its couterpart, $Z(\leq n)$, where

$$Z(\leq n) := \bigvee_{d \leq n} Z(d),$$

with $Z(d)$ is a localized version of the cochain complex resolving the comodule $A/(u_d)$, with u_d the d^{th} P^s_t element in the obvious ordering.)

We recall that finite localization away from C is the same as finite localization away from D if and only if C and D generate the same thick subcategories. But the thick subcategories generated by $HA(k)/v_n$ and $F(n+1)$ are quite different, since the times p map acts nonnilpotently on $HA(k)/v_n$ but not on $F(n+1)$.

The upshot of all this is that in $St(k)$

$$L^f_{n,k} M^\bullet_k \neq v_n^{-1} M^\bullet_k,$$

for at least one injective $A(k)_*$-resolution $M^\bullet_k$.

This does **NOT** mean that the L^f_n-Ext conjecture is false, only that the most obvious analog in the $A(k)$-world can't be true. Further, this counterexample $M^\bullet_k$ seems likely to be an infinite spectrum in $St(k)$, as we will see below.

The following Lemma shows how to use information on the $St(k)$ level to help us address the actual conjecture.

Lemma 10. *For all finite $X^\bullet \in St$,*

$$L^f_n X^\bullet = \varprojlim_k L^f_{n,k} X^\bullet_k.$$

Note that, at least on the surface, we've somehow interchanged a direct and an inverse limit, which seems woefully optimistic. However, there is a simple way to think about this: First, we show that the functor

$$L'(X^\bullet) := \varprojlim_k L^f_{n,k} X^\bullet_k$$

is a localization functor on the category St. Then we show that it agrees with L^f_n. One's initial inclination in approaching such a question might be to use some sort of thick subcategory argument, but there's no known characterization of thick subcategories of St. However, it's still easy to compare L' and L^f_n in St.

First, we need to make sure that L' is even defined. In particular, for a complex $X^\bullet \in St$, we need to know that the map given by the following diagram makes sense:

$$\begin{tikzcd} X^\bullet_{k+1} \arrow[r, "forget"] \arrow[d, "L"] & X^\bullet_k \arrow[d, "L"] \\ L^f_{n,k+1}X^\bullet_{k+1} \arrow[r, dashed, "???"] & L^f_{n,k}X^\bullet_k \end{tikzcd}$$

The localizations on the bottom row take place in different categories, so there's no obvious map, at least on the surface. However, we recall that the Mahowald-Sadofsky theorem shows that $L^f_n X^\bullet$ can be computed by taking $\varinjlim_I X^\bullet \wedge \overline{M(I)}$, for $M(I)$ appropriate finite complexes of type $n+1$. In the $St(k)$ setting, we can just choose these complexes as their images under the forgetful functor $St \to St(k)$. So it's clear that the map in question is just the forgetful functor applied both to $X^\bullet_{k+1}$ and to the $\overline{M}$s.

Recall that a functor L on a stable homotopy category is a localization if

- The natural transformation L to L^2 is an equivalence.
- $[LX, LY]_* \to [X, LY]_*$ is an iso.
- $LX = 0$ implies that $L(X \wedge Y) = 0$ for all Y.

For our functor L' above, we know that on the $St(k)$ level, all three conditions are satisfied by $L^f_{n,k}$. In particular, it's easy to see that the inverse limit of the isomorphisms $L^f_{n,k} \cong (L^f_{n,k})^2$ shows that $L' \to (L')^2$ is an equivalence. The second condition is more troublesome, because the homomorphism induced from $X^\bullet \to L'X^\bullet$,

$$[L'X^\bullet, L^\bullet Y^\bullet] \to [X', L'Y^\bullet]$$

is an inverse limit of $St(k)$ maps on the second variable and a direct limit of $St(k)$ maps on the first variable. However, our observation that $L^f_{n,k}$ can be described as smashing with a direct limit of $\overline{M}_k$s shows that

$$(L'X^\bullet)_k = L_{n,k}X^\bullet_k,$$

where, as usual, $D^\bullet_k$ is the image of the complex $D^\bullet$ under the forgetful functor, $St \to St(k)$. So the desired homomorphism

$$[L'X^\bullet, L^\bullet Y^\bullet] \to [X', L'Y^\bullet]$$

can be thought of either as the direct limit (on k) of

$$[L^f_{n,k}X^\bullet_k, (L^\bullet Y^\bullet)_k]_{St(k)} \to [X^\bullet_k, (L^\bullet Y^\bullet)_k]_{St(k)},$$

which are all isomorphisms, or as the inverse limit (on j) of

$$[(L'X^\bullet)_j, L^f_{n,j}Y^\bullet_j]_{St(j)} \to, [X^\bullet_j, L^f_{n,j}Y^\bullet_j]_{St(j)}$$

which are also all isomorphisms. Finally, since $L^f_{n,k}$ is smashing, so is the inverse limit L', so that if $L'X^\bullet = 0$, then smashing with $Y^\bullet_k$ still yields zero. Hence L is indeed a localization.

How do we show that L' agrees with L^f_n? We recall that finite localization away from C is the same as finite localization away from D if and only if C and D generate the same thick subcategories. Now L^f_n is finite localization away from $F(n+1)$, a finite complex of type $n+1$ in St. But $L' = \varprojlim_k L^f_{n,k}$ is clearly finite localization away from $F = \varprojlim_k F_k(n+1)$, an inverse limit of finite type $n+1$ in the categories $St(k)$. For appropriate choices of $F_k(n+1)$, this inverse limit is exactly an $F(n+1)$ in St. This completes the proof of the lemma.

Everyone who's thought about the L^f_n-Ext conjecture seems to have believed the following:

Conjecture 11. *(Local conjecture) For any cochain complex $X^\bullet \in St$,*

$$v_n^{-1}X^\bullet = \varprojlim_i v_n^{-1}HA(i) \wedge X^\bullet$$

is L^f_n-local in St. Similarly, for any $X^\bullet \in St(k)$, $v_n^{-1}X^\bullet$ is L^f_n-local there.

In particular, Palmieri states that the Local Conjecture is true for all finite $X^\bullet \in St$. In light of this, it's hard to see why the $St(k)$ analogue wouldn't also be true, until one gets into the guts of the proof.

Theorem 12. *The Local Conjecture (11) is true for any finite $X^\bullet \in St$.*

Proof: The easiest way to see this is to use the formulation of L_n^f as finite localization with respect to $E(n)$. (As above, in the St setting, of course, we replace the spectrum $E(n)$ (where $E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_n^{\pm 1}]$) by $Z(\leq n)$, a wedge of the periodic versions of the complexes which resolve the A-modules $A/(u_d)$, with $d \leq n$, where u_d is Palmieri's notation for the P_t^s's ordered appropriately.) Let T be any finite $E(n)$-acyclic. Consider $[T, v_n^{-1}X^\bullet]_*$. Intuitively, we think of such a finite T as being $E(n)$-acyclic in St because it has a vanishing line of slope less than or equal to $1/(|v_n|+1)$. More precisely, there must be such a vanishing line in some E_r of the Adams spectral sequence for T, so we conclude that

$$[T, v_n^{-1}X^\bullet]_* = [S^0, DT \wedge v_n^{-1}X^\bullet]_* = [S^0, v_n^{-1}(DT \wedge X^\bullet)]_* = 0,$$

where the second equality follows from the centrality of v_n-self maps. Without finiteness on T, we don't have the needed vanishing line. Further, if X were not finite, we don't have the centrality needed to conclude that the homotopy classes of maps must add up to zero.

The result is almost certainly not true for a general $X^\bullet \in St$. Also, in the $St(k)$ world, we don't have such vanishing lines, because of the h_0-towers in Ext. There seems to be no clear $St(k)$ analogue of this result.

Here's why this Local conjecture is the key point:

Lemma 13. *If $X_k^\bullet \in St(k)$ is a finite spectrum for $k > n$, then*

$$L_{n,k}^f v_n^{-1} X_k^\bullet \cong L_{n,k}^f X_k^\bullet.$$

Further, the obvious analogue holds in St itself:

$$L_n^f v_n^{-1} X^\bullet \cong L_n^f X^\bullet.$$

Proof. For finite $X_k^\bullet \in St(k)$,

$$v_n^{-1}X_k^\bullet = v_n^{-1}HA(k) \wedge X_k^\bullet = v_n^{-1}G(k)^\bullet \wedge X_k^\bullet,$$

is just a telescope. Since the maps in the telescope are just multiplication by some power of v_n, these maps are just the identity in $L_n^f X_k^\bullet$. Thus the finite localization map factors through the telescope, as usual:

$$\begin{tikzcd} X_k^\bullet \arrow[r] \arrow[dr] & v_n^{-1}X_k^\bullet \arrow[d] \\ & L_{n,k}^f X_k^\bullet \end{tikzcd}$$

and clearly $L^f_{n,k} v_n^{-1} X^\bullet_k \cong L^f_{n,k} X^\bullet_k$ in $St(k)$. If $X^\bullet \in St$, taking an inverse limit of these diagrams for $X^\bullet_k$ and using Lemma 7 completes the proof.

Note that this lemma, along with the Local theorem above, implies the following: for all finite $X^\bullet \in St$, the localization map

$$v_n^{-1} X^\bullet \xrightarrow{loc} L^f_n X^\bullet,$$

is an isomorphism in St, since

$$v_n^{-1} X^\bullet = L^f_n v_n^{-1} X^\bullet = L^f_n X^\bullet,$$

where the first equality follows from the Local Conjecture and the second from Lemma 13. However, if this were true in the $St(k)$ category for all $X^\bullet \in St(k)$, this would directly contradict Hovey's observation.

References

1. Bajer, A., and Sadofsky, H., Cohomology of finite dimensional, connected, cocommutative Hopf algebras, *J. Pure Appl. Alg.* **94** (1994) pp 115-126.
2. Davis, D.M., Mahowald, M. E., and Miller, H.R., Mapping telescopes and K_*-localization, *Algebraic Topology and Algebraic K-theory,* ed. by W. Browder, Princeton Annals Studies 113 (1987) pp 152-167.
3. Lin, W.-H., Cohomology of sub-Hopf algebras of the Steenrod algebra, *J. Pure Appl. Alg.* **10** (1977) pp 101-114.
4. Hopkins, M., and Smith, J., Nilpotence in stable homotopy theory II, *Annals of Math.* **148** (1998) pp 1–49.
5. Hovey, M., Palmieri, J., and Strickland, N., Axiomatic stable homotopy theory, *Memoirs of the A.M.S.*, to appear.
6. Mahowald, M.E., Ravenel, D.C., and Shick, P., The triple loop space approach to the Telescope Conjecture, *Homotopy Methods in Algebraic Topology, Boulder, 1999*, ed. by J.P.C. Greenlees, Amer. Math Soc., to appear.
7. Mahowald, M.E., and Sadofsky, H., v_n-telescopes and the Adams spectral sequence, *Duke J. Math.* **78** (1995) pp 101-129.
8. Mahowald, M.E., and Shick, P., Periodic phenomena in the classical Adams spectral sequence, *Trans. A.M.S.* **300** (1987) pp 191-206.
9. Miller, H.R., Finite localizations, *Boletin de la Sociedad Mat. Mex.* **37** (J. Adem Memorial volume)(1992) pp 383-390.
10. Palmieri, J., The chromatic filtration and the Steenrod algebra, *Topology and Representation Theory*, ed. by E. Friedlander and M. Mahowald, *Contemp. Math* **158** (1994) pp187-201.
11. Palmieri, J., Stable homotopy over the Steenrod algebra, *Memiors of the A.M.S.*, to appear.
12. Palmieri, J., and Sadofsky, H., Self maps of spectra, a theorem of J. Smith, and the Margolis killing construction, *Math. Zeit.* **215** (1994) pp 477-490.
13. Ravenel, D.C., Localization with respect to certain periodic homology theories, *Amer. J. Math.*, **106** (1984) pp 351–414.

14. Ravenel, D.C., Some variations on the telescope conjecture, *The Cech Centennial*, ed. by M. Cenkl and H. Miller, *Contemp. Math.* **181** (1995) pp391-403.
15. Shick, P., Odd primary periodic phenomena in the classical Adams spectral sequence, *Trans. A.M.S.*, **309** (1988) pp 77-86.
16. Wilkerson, C., The cohomology algebras of finite dimensional Hopf algebras, *Trans. A.M.S.*, **264** (1981) pp 137-150.

John Carroll University, University Heights, OH 44118