# The Thomified Eilenberg-Moore spectral sequence

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## 1. Introduction

In this paper we will construct a generalization of the Eilenberg-Moore spectral sequence, which in some interesting cases turns out to be a form of the Adams spectral sequence. We recall the construction of both of these in general terms. Suppose we have a diagram of spectra of the form

$$\begin{tikzcd} X_0 \arrow[d, "g_0"'] & X_1 \arrow[l] \arrow[d, "g_1"'] & X_2 \arrow[l] \arrow[d, "g_2"'] & \cdots \arrow[l] \\ K_0 & K_1 & K_2 & \end{tikzcd} \tag{1.1}$$

where $X_{s+1}$ is the fiber of $g_s$. We get an exact couple of homotopy groups and a spectral sequence with

$$E_1^{s,t} = \pi_{t-s}(K_s) \quad \text{and} \quad d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}.$$

This spectral sequence converges to $\pi_*(X)$ (where $X = X_0$) if the homotopy inverse limit $\lim_{\leftarrow} X_s$ is contractible and certain $\lim^1$ groups vanish. When $X$ is connective, it is a first quadrant spectral sequence. For more background, see [Rav86].

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In the case of the classical Adams spectral sequence, we have some additional conditions on on (1.1), namely

- Each spectrum $K_s$ is a generalized mod $p$ Eilenberg-Mac Lane spectrum, and
- each map $g_s$ induces a monomorphism in mod $p$ homology

These conditions enable us to identify the $E_2$-term as an Ext group over the Steenrod algebra, and to prove convergence when $X$ is connective and $p$-adically complete.

For the Eilenberg-Moore spectral sequence, let

$$X \xrightarrow{i} E \xrightarrow{h} B \tag{1.2}$$

be a fiber sequence with simply connected base space $B$. Then one uses this (in a manner to be described below) to produce a diagram of the form (1.1) where $X_0$ is the suspension spectrum of $X$. This will yield a spectral sequence converging to the stable homotopy of $X$, but in practice it is not very useful. However if we smash everything in sight with the mod $p$ Eilenberg-Mac Lane spectrum $H/p$, we get the Eilenberg-Moore spectral sequence converging to $H_*(X)$, where $E_2$ is a certain Cotor group over $H_*(B)$.

In this paper we will explain a way to twist this construction using a $p$-local spherical fibration over the total space $E$. The entire construction can be Thomified to yield a spectral sequence converging to the homotopy of the Thom spectrum for the induced bundle over $X$. In §2 we recall a geometric construction of the Eilenberg-Moore spectral sequence, and in §3 we explain how it can be Thomified. In §4 we identify the $E_2$-term under certain circumstances as an Ext group over the Massey-Peterson algebra of the base space of the fibration in question, and in §5 we show that in some other cases we get a $BP$-theoretic analog of this result. In §6, we show that a special case of the $\mathbf{Z}/(p)$-equivariant Adams spectral sequence of Greenlees can be constructed using the Thomified Eilenberg-Moore spectral sequence.

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## 2. A geometric construction of the Eilenberg-Moore spectral sequence

We begin by recalling the stable cosimplicial construction associated with the Eilenberg-Moore spectral sequence, due to Larry Smith [Smi69] and Rector [Rec70]. Given the fibration (1.2), for $s \geq 0$ let

$$G_s = E \times \overbrace{B \times \cdots \times B}^{s \text{ factors}}.$$

Define maps $h_t : G_{s-1} \to G_s$ for $0 \leq t \leq s$ by

$$h_t(e, b_1, b_2, \ldots, b_{s-1}) = \begin{cases} (e, b_1, b_2, \ldots, b_{s-1}, *) & \text{if } t = 0 \\ (e, b_1, b_2, \ldots, b_t, b_t, b_{t+1}, \ldots, b_{s-1}) & \text{if } 1 \leq t \leq s-1 \\ (e, h(e), b_1, b_2, \ldots, b_{s-1}) & \text{if } t = s. \end{cases}$$

Let $E_0 = E$, $X_0 = X$, $X_1 = E/\operatorname{Im} i$, and for $s \geq 1$ we define spectra

$$\begin{aligned} \Sigma^s E_s &= G_s/\operatorname{Im} h_0 \cup \cdots \cup \operatorname{Im} h_{s-1} \\ \Sigma^{s+1} X_{s+1} &= G_s/\operatorname{Im} h_0 \cup \cdots \cup \operatorname{Im} h_s \end{aligned}$$

i.e., the spectra $X_s$ and $E_s$ are desuspensions of suspension spectra of the indicated spaces. Then for $s \geq 0$, $h_s$ induces a map $X_s \to E_s$ giving a cofiber sequence

$$X_s \xrightarrow{h_s} E_s \xrightarrow{\partial_s} \Sigma X_{s+1}, \tag{2.1}$$

where $\partial_s$ is projection from the topological quotient of $G_s$ by one subspace to the quotient by a bigger subspace.

For $s \geq 0$ there is a homology isomorphism

$$H_*(E_s) = \Sigma^{-s} H_*(E) \otimes \overline{H}_*(B^{(s)}),$$

where $\overline{H}$ denotes reduced homology. Since $B$ is simply connected, the connectivity of $E_s$ increases without bound with $s$. Note also that

$$H_*(X_s) = \Sigma^{-s} H_*(X) \otimes \overline{H}_*(B^{(s)}),$$

for $s > 0$, so the homotopy inverse limit of the $X_s$ is contractible. The homology exact couple associated with the cofiber sequences (2.1) leads to the Eilenberg-Moore spectral sequence for the fibration (1.2).

The Eilenberg-Moore spectral sequence also converges for non-simply-connected $B$, also. Dwyer has proved ([D74]) that the Eilenberg-Moore spectral sequence for the fibration

$$X \to E \to B$$

converges strongly to $H_*(X)$ if and only if $\pi_1(B)$ acts nilpotently on $H_i(E)$ for all $i \geq 0$.

# 3. The Thomified Eilenberg-Moore spectral sequence

Now suppose that in addition to the fibration (1.2) we also have a $p$-local stable spherical fibration $\xi$ over $E$ which is oriented with respect to mod $p$ homology. Projection onto the first coordinate gives compatible maps of the $G_s$ to $E$, and hence a stable spherical fibration over each of them. This means that we can Thomify the entire construction. To each of the quotients $X_s$ and $E_s$ we associate a *reduced Thom spectrum*, which is defined as follows. Given a space $A$ with a spherical fibration and a subspace $B \subset A$, the reduced Thom space for $A/B$ is the space $D_A/(S_A \cup D_B)$ where $D_X$ and $S_X$ denote disk and sphere bundles over the space $X$. Thus we can associate reduced Thom spectra to the topological quotients $E_s$ and $X_{s+1}$ of $G_s$.

Let $Y$, $K$, $Y_s$ and $K_s$ be the Thomifications of $X$, $E$, $X_s$ and $E_s$. Then the cofiber sequence of (2.1) Thomifies to

$$Y_s \longrightarrow K_s \longrightarrow \Sigma Y_{s+1}. \tag{3.1}$$

and we have

$$H_*(K_s) = \Sigma^{-s} H_*(K) \otimes \overline{H}_*(B^{(s)}).$$

The exact couple of homotopy groups for (3.1) leads to a spectral sequence converging to $\pi_*(Y)$. There is an associated diagram

$$\begin{tikzcd} Y \arrow[r, equal] & Y_0 \arrow[d, "g_0"'] & Y_1 \arrow[l] \arrow[d, "g_1"'] & Y_2 \arrow[l] \arrow[d, "g_2"'] & \cdots \arrow[l] \\ & K_0 & K_1 & K_2 & \end{tikzcd} \tag{3.2}$$

where $Y_{s+1}$ is the fiber of $g_s$. This is similar to the Adams diagram of (1.1), but $H_*(g_s)$ need not be a monomorphism in general. We will call this the *Thomified Eilenberg-Moore spectral sequence.* We will use the indexing conventions of Adams rather than Eilenberg-Moore, namely

$$E_1^{s,t} = \pi_{t-s}(K_s) \qquad \text{with} \qquad E_r^{s,t} \xrightarrow{d_r} E_r^{s+r,t+r-1}.$$

This puts our spectral sequence in the first rather than the second quadrant.

We will see below (Theorem 4.4(ii) and Corollary 4.5) that under suitable hypotheses (including that the map $i$ of (1.2) induces a monomorphism in homology), the Thomified Eilenberg-Moore spectral sequence coincides with the usual Adams spectral sequence for $\pi_*(Y)$.

The following lemma will be useful.

**Lemma 3.3.** *For each prime $p$ there is a $p$-local spherical fibration over $\Omega^2 S^3$ whose Thom spectrum is the mod $p$ Eilenberg-Mac Lane spectrum $H/p$.*

*Proof.* For $p = 2$ we can use an ordinary vector bundle. We extend the nontrivial map $S^1 \to BO$ to $\Omega^2 S^3$ using the double loop space structure on $BO$. It was shown in [Mah79] that the resulting Thom spectrum is $H/2$.

The following argument for odd primes is due to Mike Hopkins. Let $BF(n)_{(p)}$ denote the classifying space for the monoid of homotopy equivalences of the $p$-local $n$-sphere. Its fundamental group is $\mathbf{Z}_{(p)}^{\times}$. A $p$-local $n$-dimensional spherical fibration of a space $X$, i.e., a fibration with fiber $S_{(p)}^n$, is classified by a map $X \to BF(n)_{(p)}$. Its Thom space is the cofiber of the projection map to $X$. Such fibrations and Thom spectra can be stabilized in the usual way. We denote the direct limit of the $BF(n)_{(p)}$ by $BF_{(p)}$.

Now consider a $p$-local spherical fibration over $S^1$ corresponding to an element $u \in \mathbf{Z}_{(p)}^{\times}$. It Thomifies to the Moore spectrum $S^0 \cup_{1-u} e^1$. If we set $u = 1 - p$ (which is a $p$-local unit) we get the mod $p$ Moore spectrum $V(0)$.

As in the case $p = 2$, we can extend this map $S^1 \to BF_{(p)}$ to $\Omega^2 S^3$ using the double loop space structure on $BF_{(p)}$, and similar arguments to those of [Mah79] identify the resulting Thom spectrum as $H/p$. $\square$

# 4. Identifying the $E_2$-term

Observe that $H_*(K)$ is simultaneously a comodule over $A_*$ and (via the Thom isomorphism and the map $h_*$) $H_*(B)$, which is itself a comodule over $A_*$. Following Massey-Peterson [MP67], we combine these two structures by defining the *Massey-Peterson coalgebra* (they called the dual object the semitensor product)

$$R_* = H_*(B) \otimes A_* \tag{4.1}$$

in which the coproduct is the composite

$$\begin{array}{c}
H_*(B) \otimes A_* \\
\Big\downarrow {\scriptstyle \Delta_B \otimes \Delta_A} \\
H_*(B) \otimes H_*(B) \otimes A_* \otimes A_* \\
\Big\downarrow {\scriptstyle H_*(B)\otimes\psi_B\otimes A_*\otimes A_*} \\
H_*(B) \otimes A_* \otimes H_*(B) \otimes A_* \otimes A_* \\
\Big\downarrow {\scriptstyle H_*(B)\otimes A_*\otimes T\otimes A_*} \\
H_*(B) \otimes A_* \otimes A_* \otimes H_*(B) \otimes A_* \\
\Big\downarrow {\scriptstyle H_*(B)\otimes m_A\otimes H_*(B)\otimes A_*} \\
(H_*(B) \otimes A_*) \otimes (H_*(B) \otimes A_*),
\end{array} \tag{4.2}$$

where $\Delta_A$ and $\Delta_B$ are the coproducts on $A_*$ and $H_*(B)$, $T$ is the switching map, $\psi_B : H_*(B) \to A_* \otimes H_*(B)$ is the comodule structure map, and $m_A$ is the multiplication in $A_*$.

Massey-Peterson gave this definition in cohomological terms. They denoted the semitensor algebra $R$ by $H^*(B) \odot A$, which is additively isomorphic to $H^*(B) \otimes A$ with multiplication given by

$$(x_1 \otimes a_1)(x_2 \otimes a_2) = x_1 a_1'(x_2) \otimes a_1'' a_2,$$

where $x_i \in H^*(B)$, $a_i \in A$, and $a_1' \otimes a_1''$ denotes the coproduct expansion of $a_1$ given by the Cartan formula. Our definition is the homological reformulation of theirs.

Note that given a map $f : V \to B$ and a subspace $U \subset V$, $\bar{H}^*(V/U) = H^*(V, U)$ is an $R$-module since it is an $H^*(V)$-module via relative cup products, even if the map $f$ does not extend to the quotient $V/U$. In our case we have maps $G_s \to B$ for all $s \geq 0$ given by

$$(e, b_1, \ldots, b_s) \mapsto h_e.$$

These are compatible with all of the maps $h_t$, so $H_*(Y_s)$ and $H_*(K_s)$ are $R_*$-comodules, and the maps between them respect this structure.

We will see in the next theorem that under suitable hypotheses, the $E_2$-term of the Thomified Eilenberg-Moore spectral sequence is $\mathrm{Ext}_{R_*}(\mathbf{Z}/(p), H_*(K))$. When $B$ is an H-space we have a Hopf algebra extension (see [Rav86, A1.1.15] for a definition)

$$A_* \longrightarrow R_* \longrightarrow H_*(B).$$

This gives us a Cartan-Eilenberg spectral sequence ([CE56, page 349] or [Rav86, A1.3.14]) converging to this Ext group with

$$E_2 = \mathrm{Ext}_{A_*}(\mathbf{Z}/(p), \mathrm{Ext}_{H_*(B)}(\mathbf{Z}/(p), H_*(K))). \tag{4.3}$$

Note that the inner Ext group above is the same as $\mathrm{Ext}_{H_*(B)}(\mathbf{Z}/(p), H_*(E))$, the $E_2$-term of the classical Eilenberg-Moore spectral sequence converging to $H_*(X)$. If the latter collapses from $E_2$, then the Ext group of (4.3) can be thought of as

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/(p), H_*(Y)),$$

where $H_*(Y)$ is equipped with the Eilenberg-Moore bigrading. This is the usual Adams $E_2$-term for $Y$ when $H_*(Y)$ is concentrated in Eilenberg-Moore degree 0, but the Ext group of (4.3) is graded differently in general.

**Theorem 4.4.** (i) *Suppose that $B$ is simply connected. Then the Thomified Eilenberg-Moore spectral sequence associated with the homotopy of (3.2) converges to $\pi_*(Y)$. If, in addition, $H^*(K)$ is a free $A$-module, then*

$$E_2 = \mathrm{Ext}_{R_*}(\mathbf{Z}/(p), H_*(K)),$$

*where $R_*$ is the Massey-Peterson coalgebra of (4.1).*

(ii) *If, in addition, the map $i : X \to E$ induces a monomorphism in mod $p$ homology, then the Thomified Eilenberg-Moore spectral sequence coincides with the classical Adams spectral sequence for $Y$.*

The hypotheses on $H_*(K)$ may be unnecessary, but they are adequate for our purposes. The result may not be new, but we know of no published proof. Before proving the theorem we give a corollary that indicates that the hypotheses are not as restrictive as they may appear.

**Corollary 4.5.** *Given a fibration*

$$X \longrightarrow E \longrightarrow B$$

*with $X$ $p$-adically complete, a $p$-local spherical fibration over $E$, and $B$ simply connected, there is a spectral sequence converging to $\pi_*(Y)$ (where $Y$ is the Thomification of $X$) with*

$$E_2 = \mathrm{Ext}_{H_*(B)\otimes A_*}(\mathbf{Z}/(p), H_*(K)),$$

*where $K$ as usual is the Thomification of $E$.*

*Proof.* We can apply 4.4 to the product of the given fibration with pt. $\to \Omega^2 S^3 \to \Omega^2 S^3$, where $\Omega^2 S^3$ is equipped with the $p$-local spherical fibration of Lemma 3.3. Then the Thomified total space is $K \wedge H/p$, so its cohomology is a free $A$-module. Thus the $E_2$-term is

$$\mathrm{Ext}_{H_*(B\wedge H/p)\otimes A_*}(\mathbf{Z}/(p), H_*(K \wedge H/p)) = \mathrm{Ext}_{H_*(B)\otimes A_*}(\mathbf{Z}/(p), H_*(K)).$$

□

*Proof of Theorem 4.4* (i) The freeness of $H_*(K)$ over $A_*$ does not make (3.2) an Adams resolution because $H_*(g_s)$ need not be a monomorphism and the cofiber sequence

$$\Sigma^s Y_s \xrightarrow{g_s} \Sigma^s K_s \longrightarrow \Sigma^{s+1} Y_{s+1}$$

need not induce a short exact sequence in homology.

We will finesse this problem by producing a commutative diagram

$$\begin{tikzcd} \Sigma^s Y_s \arrow[r, "g_s"] \arrow[d, "g_s"] & \Sigma^s K_s \arrow[r] \arrow[d] & \Sigma^{s+1} Y_{s+1} \arrow[d, "h_{s+1}"] \\ \Sigma^s K_s \arrow[r] & \Sigma^s W_s \arrow[r] & \Sigma^{s+1} K_{s+1} \end{tikzcd} \quad \text{for } s \geq 0. \tag{4.6}$$

in which the cofiber sequence in the bottom row does induce a short exact sequence in homology with

$$H_*(W_s) = H_*(K_s) \otimes H_*(B). \tag{4.7}$$

By the change-of-rings isomorphism of Milnor-Moore [MM65], this implies that

$$\mathrm{Ext}_{R_*}(\mathbf{Z}/(p), H_*(W_s)) = \mathrm{Ext}_{A_*}(\mathbf{Z}/(p), H_*(K_s)). \tag{4.8}$$

Splicing the short exact sequences in homology from the bottom row of (4.6) gives a long exact sequence

$$0 \longrightarrow H_*(K) \longrightarrow H_*(W_0) \longrightarrow H_*(\Sigma W_1) \longrightarrow \cdots,$$

which gives an algebraic spectral sequence (see [Rav86, A1.3.2]) converging to $\mathrm{Ext}_{R_*}(\mathbf{Z}/(p), H_*(K))$ with

$$E_1 = \mathrm{Ext}_{R_*}(\mathbf{Z}/(p), H_*(W_s)),$$

suitably indexed.

The freeness hypothesis on $H_*(K)$ implies (via (4.7)) that $H_*(W_s)$ is free over $R_*$, so the algebraic spectral sequence collapses from $E_2$, i.e., $\mathrm{Ext}_{R_*}(\mathbf{Z}/(p), H_*(K))$ is the cohomology of the cochain complex

$$\mathrm{Ext}^0_{R_*}(\mathbf{Z}/(p), H_*(W_0)) \longrightarrow \mathrm{Ext}^0_{R_*}(\mathbf{Z}/(p), H_*(\Sigma W_1)) \longrightarrow \cdots$$

By (4.8) this is the same as

$$\mathrm{Ext}^0_{A_*}(\mathbf{Z}/(p), H_*(K_0)) \longrightarrow \mathrm{Ext}^0_{A_*}(\mathbf{Z}/(p), H_*(\Sigma K_1)) \longrightarrow \cdots$$

and our freeness hypothesis along with (4.6) allows us to identify this cochain complex with the $E_1$-term of the Thomified Eilenberg-Moore spectral sequence.

Thus the Thomified Eilenberg-Moore spectral sequence has the desired $E_2$-term if we can produce the diagram (4.6) satisfying (4.7). We shall do this now by geometric construction.

We define the following subspaces of $G_s$ for $s \geq 1$:

$$\begin{aligned} A_s &= \operatorname{Im} h_0 \cup \operatorname{Im} h_2 \cup \ldots \cup \operatorname{Im} h_{s-1}, \\ B_s &= A_s \cup \operatorname{Im} h_1 \\ \text{and} \quad C_s &= B_s \cup \operatorname{Im} h_s. \end{aligned}$$

Then it follows that $h_s$ sends $C_{s-1}$ to $B_s$ and $B_{s-1}$ to $A_s$, $B_s/A_s = G_{s-1}/B_{s-1}$ and $C_s/B_s = G_{s-1}/C_{s-1}$. Thus for $s \geq 0$ we get the following pointwise commutative diagram in which each row is a cofiber sequence.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "i"] & E \arrow[r, "\partial_0"] \arrow[d, "h_1"] & E/i(X) \arrow[d, "h_1"] \\ E \arrow[r, "h_1"] & G_1/A_1 \arrow[r, "\partial_1"] & G_1/B_1 \end{tikzcd} \qquad \text{for } s = 0$$

$$\text{and} \qquad \begin{tikzcd} G_{s-1}/C_{s-1} \arrow[r, "h_s"] \arrow[d, "h_s"] & G_s/B_s \arrow[r, "\partial_s"] \arrow[d, "h_{s+1}"] & G_s/C_s \arrow[d, "h_{s+1}"] \\ G_s/B_s \arrow[r, "h_{s+1}"] & G_{s+1}/A_{s+1} \arrow[r, "\partial_{s+1}"] & G_{s+1}/B_{s+1} \end{tikzcd} \qquad \text{for } s \geq 1.$$

We define $\Sigma^{s-1}W_{s-1}$ to be the Thomification of $G_s/A_s$, and we have previously defined $\Sigma^s K_s$ and $\Sigma^{s+1}X_{s+1}$ to be the Thomifications of $G_s/B_s$ and $G_s/C_s$, so Thomification converts the diagrams above to (4.6).

Let $p_s : G_{s+1} \to G_s \times B$ be the homeomorphism given by

$$p_s(e, b_1, \ldots, b_{s+1}) = ((e, b_2, \ldots, b_{s+1}), b_1).$$

Then we have

$$\begin{aligned} p_s h_0 &= (h_0 \times B) p_{s-1} \\ \text{and} \quad p_s h_t &= (h_{t-1} \times B) p_{s-1} \qquad \text{for } 2 \leq t \leq s. \end{aligned}$$

It follows that

$$G_{s+1}/A_{s+1} = (G_s \times B)/(B_s \times B) = (G_s/B_s) \times B$$

and (4.7) follows.

(ii) If the $H_*(i)$ is monomorphic and $H^*(K_s)$ is a free $A$-module, then the diagram (3.2) is an Adams resolution for $Y$. Thus, the identity map on the resolution provides a comparison map from the Thomified Eilenberg-Moore spectral sequence to the Adams spectral sequence. We can identify the inner Ext group of (4.3) with $H_*(Y)$ concentrated in degree 0, the Cartan-Eilenberg spectral sequence collapses and our $E_2$-term is the usual

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/(p), H_*(Y)).$$

So the comparison map induces an isomorphism on the $E_2$ term of the spectral sequences, completing the proof of the theorem.

□

## 5. An Adams-Novikov analog

We now describe a case of the Thomified Eilenberg-Moore spectral sequence leading to variants of the Adams-Novikov spectral sequence. Suppose that in the fibration of (1.2), the spherical fibration over $E$ is a complex vector bundle and that $MU_*(K)$ is free as a comodule over $MU_*(MU)$. If in addition $MU_*(i)$ is a monomorphism, then we get the usual Adams-Novikov spectral sequence converging to $\pi_*(Y)$.

We want an analog of 4.4 in the $p$-local case identifying the $E_2$-term for more general $i$. For this we need a BP-theoretic analog of the Massey-Peterson algebra $R_*$ of (4.1), additively isomorphic to

$$\Gamma(B) = BP_*(B) \otimes_{BP_*} \Gamma, \tag{5.1}$$

where $\Gamma = BP_*(BP)$. In order to define a coproduct on this as in (4.2), we need a coalgebra structure on $BP_*(B)$. This does not exist in general, but it does when $H_*(B)$ is torsion free and $BP_*(B)$ is therefore a free $BP_*$-module. If $B$ is also an H-space, then $BP_*(B)$ is a Hopf algebra over $BP_*$ and $(BP_*, \Gamma(B))$ is a Hopf algbebroid (defined in [Rav86, A1.1.1])

$$(BP_*, \Gamma) \longrightarrow (BP_*, \Gamma(B)) \longrightarrow (BP_*, BP_*(B))$$

is a Hopf algebroid extension as defined in [Rav86, A1.1.15]. This means there is a Cartan-Eilenberg spectral sequence (see [CE56, page 349] or [Rav86, A1.3.14]) converging to $\mathrm{Ext}_{\Gamma(B)}(BP_*, BP_*(K))$ with

$$E_2 = \mathrm{Ext}_\Gamma(BP_*, \mathrm{Ext}_{BP_*(B)}(BP_*, BP_*(K))). \tag{5.2}$$

Then we get the following analog of Theorem 4.4, which can be proved in the same way.

**Theorem 5.3.** (i) *Suppose that $BP_*(K)$ is free as a $BP_*(BP)$-comodule and $B$ is simply connected with torsion free homology. Then the Thomified Eilenberg-Moore spectral sequence associated with the homotopy of (3.2) converges to $\pi_*(Y)$ with*

$$E_2 = \mathrm{Ext}_{\Gamma(B)}(BP_*, BP_*(K)),$$

*where $\Gamma(B)$ is the Massey-Peterson coalgebra of (5.1).*

(ii) *If in addition the map $i : X \to E$ induces a monomorphism in BP-homology, then the Thomified Eilenberg-Moore spectral sequence coincides with the Adams-Novikov spectral sequence for $Y$.*

There is an analog of 4.5 in which we retain the hypothesis on $B$ while dropping the one on $K$.

**Corollary 5.4.** *Given a fibration*

$$X \longrightarrow E \longrightarrow B$$

*with $X$ p-local, a a complex vector bundle over $E$, and $B$ simply connected with torsion free homology, there is a spectral sequence converging to $\pi_*(Y)$ (where $Y$ is the Thomification of $X$) with*

$$E_2 = \mathrm{Ext}_{\Gamma(B)}(BP_*, BP_*(K)),$$

*where $K$ as usual is the Thomification of $E$.*

This can be proved by applying 5.3 to the product of the given fibration with

$$\mathrm{pt.} \longrightarrow BU \longrightarrow BU$$

with the universal complex vector bundle over $BU$.

## 6. A construction of the equivariant Adams spectral sequence

In this section we provide an alternative construction of a special case of the equivariant Adams spectral sequence, due to Greenlees ([G88] and [G90].) We first recall Greenlees' approach.

Let $G$ be a finite $p$-group. (Later, we will restrict our attention to the case where $G$ is elementary abelian.) We work in the equivariant stable homotopy category of [LMS86], with all spaces pointed and all homology groups reduced. In this setting, $G$-free means that the action of $G$ is free away from the base point. Greenlees' version of the equivariant Adams spectral sequence is based on mod $p$ Borel cohomology, defined for a based $G$-spectrum $X$ as

$$b_G^*(X) = H^*(EG_+ \wedge_G X; \mathbf{Z}/(p)),$$

where, as above, the $\mathbf{Z}/(p)$ coefficient groups will hereafter be suppressed. This is an $RO(G)$-graded cohomology theory, defined as follows for $\alpha$ any virtual real representation of $G$:

$$b_G^\alpha(X) = H^{|\alpha|}(EG_+ \wedge_G X).$$

Since $G$ is a $p$-group, all representations are orientable, and the suspension isomorphisms in $b_G^*$ are given by the Thom maps, so the theory is really $\mathbf{Z}$-graded in this case. This cohomology theory $b_G$ is representable in the equivariant stable category. Adams and Greenlees identify the algebra $b_G^*(b_G)$ of natural cohomology operations as

$$b_G^*(b_G) \cong H^*(BG_+)\tilde{\otimes} A,$$

where $\tilde{\otimes}$ denotes the Massey-Peterson semitensor product. Greenlees actually defines the spectral sequence in terms of a variant of Borel cohomology, namely $f$- or coBorel-cohomology, represented by

$$c_G = b_G \wedge EG_+.$$

Greenlees shows in [G88] that $c_G^*(c_G) \cong b_G^*(b_G)$.

Greenlees' main result is the following cohomology version of the spectral sequence.

**Theorem 6.1.** *([G88]) For $G$ a finite $p$-group, $X$ and $Y$ any $G$-spectra, with $Y$ $p$-complete, bounded below, $G$-free and homologically locally finite, there is a convergent Adams spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{c_G^*(c_G)}(c_G^* Y, c_G^* X) \Longrightarrow [X, Y]_*^G,$$

*natural in both variables.*

One can define a similar spectral sequence based on $b_G^*(\cdot)$, but this requires the additional hypothesis that $X$ is $G$-free to guarantee proper convergence. A homology version of the spectral sequence can be written using the homology theory represented by the $G$-spectrum $b_G$, ([G90]) which does calculate $[X, Y]_*^G$ when $X$ or $Y$ is not $G$-free, provided we take $G$ to be elementary abelian. The hypotheses on $Y$ can just be checked nonequivariantly, if $Y$ is $G$-free, by looking at the non-equivariant spectrum $EG_+ \wedge_G Y$.

Greenlees' construction involves building a resolution of $b_G^* Y$ by free $b_G^*(b_G)$-modules,

$$0 \longleftarrow b^* Y \xleftarrow{\epsilon} P_0 \xleftarrow{\delta_0} P_1 \xleftarrow{\delta_1} P_1 \longleftarrow \cdots$$

and realizing this resolution geometrically. Apply the functor $[X, -]^G$ to this geometric resolution, obtaining a spectral sequence with

$$E_1 = [\Sigma^{t-s} X, Q_s]^G \Longrightarrow [\Sigma^{t-s} X, Y/\operatorname*{holim}_s Y_s]^G,$$

where $Q_s$ is a locally finite wedge of copies of the spectrum representing $b_G$ made free (i.e. a wedge of copies of $c_G = b_G \wedge EG_+$,) with $P_s = b_G^* \Sigma^s Q_s$. One identifies the $E_2$ term in the usual manner, and proves convergence by comparing $c_G^*$- (or $b_G^*$-) connectivity with $H^*$-connectivity to show that $\mathrm{holim}_s Y_s \simeq *$.

We now show how to identify this equivariant Adams spectral sequence as a case of the Thomified Eilenberg-Moore spectral sequence, with certain restrictive hypotheses. From here onward we take $G$ to be $\mathbf{Z}/(p)$, and we'll work with the spectrum $X$ $G$-fixed (so that we'll use the $c_G^*(c_G)$-based spectral sequence, rather than it $b_G^*(b_G)$-based analog.) Let $Z$ be a $p$-complete free $G$-spectrum with a spherical $G$-fibration $F \to E(\xi) \xrightarrow{p} Z$. Consider the Borel fibration

$$Z \to EG_+ \wedge_G Z \to BG.$$

The spherical $G$-fibration over $Z$ induces a $G$-fibration

$$EG_+ \wedge_G F \to EG_+ \wedge_G E(\xi) \to EG_+ \wedge_G Z,$$

so we have the desired fibration over the total space of the Borel fibration.

We smash the Borel fibration with

$$\mathrm{pt.} \to \Omega^2 S^3 \to \Omega^2 S^3$$

(with the trivial $G$-action) and apply the Thomified Eilenberg-Moore spectral sequence construction to the resulting fibration. The resulting resolution is $EG_+ \wedge_G H\mathbf{Z}/(p)$-free. Now for a $G$-fixed spectrum $W$ (like $H\mathbf{Z}/(p)$ here,) the Borel construction is very simple: $EG_+ \wedge_G W \simeq BG_+ \wedge W$, So the Thomified Eilenberg-Moore spectral sequence resolution is free over $BG_+ \wedge H\mathbf{Z}/(p)$. Let $T(Z)$ denote the Thom spectrum of the bundle over $Z$. Then the resulting Thomified Eilenberg-Moore spectral sequence has

$$\begin{aligned}
E_2 &= \mathrm{Ext}_{H_*(BG_+)\tilde{\otimes} A_*}(H_*BG_+, H_*(T(EG_+ \wedge_G Z))) \\
&= \mathrm{Ext}_{H_*(BG_+)\tilde{\otimes} A_*}(H_*BG_+, H_*(EG_+ \wedge_G T(Z))) \\
&= \mathrm{Ext}_{b_G^*(b_G)}(b_G^*(T(Z)), b_G^*(S^0)),
\end{aligned}$$

by $\mathbf{F}_p$-duality, so that the Thomified Eilenberg-Moore spectral sequence $E_2$ agrees with the equivariant Adams spectral sequence $E_2$ term.

The Thomified Eilenberg-Moore spectral sequence $E_1$ term here is given by applying the (nonequivariant) functor $\pi_*(-)$ to the Thomified Eilenberg-Moore spectral sequence diagram

$$\begin{tikzcd}
Y \arrow[r, equal] & Y_0 \arrow[d, "g_0"'] & Y_1 \arrow[l] \arrow[d, "g_1"'] & Y_2 \arrow[l] \arrow[d, "g_2"'] & \cdots \arrow[l] \\
& K_0 & K_1 & K_2 &
\end{tikzcd}$$

where $Y$ is the Thom spectrum of $EG_+ \wedge_G Z \wedge \Omega^2 S^3$. The equivariant $b_G^*(b_G)$-Adams spectral sequence $E_1$ term arises from applying $[S^0, -]_*^G$

to the geometric resolution

$$\begin{tikzcd} W \arrow[r, equal] & W_0 \arrow[d, "g_0"'] & W_1 \arrow[l] \arrow[d, "g_1"'] & W_2 \arrow[l] \arrow[d, "g_2"'] & \cdots \arrow[l] \\ & Q_0 & Q_1 & Q_2 & \end{tikzcd}$$

where $W$ is the Thom spectrum of $Z$ and $Q_s$ is a wedge of copies of the coBorel spectrum $c_G$. But the Adams isomorphism ([Ad84], 5.3) shows that

$$[S^0, Q_s]^G = [S^0, K_s]^1,$$

so that the isomorphism of $E_2$ terms above is induced by one on the $E_1$ level. This proves the following.

**Theorem 6.2.** *Let $Z$ be a $p$-complete based free $\mathbf{Z}/(p)$-spectrum, with a spherical $\mathbf{Z}/(p)$-fibration over $Z$. The Thomified Eilenberg-Moore spectral sequence for the smash product of the fibrations*

$$pt. \to \Omega^2 S^3 \to \Omega^2 S^3$$

*and*

$$Z \to E\mathbf{Z}/(p)_+ \wedge_{\mathbf{Z}/(p)} Z \to B\mathbf{Z}/(p)$$

*agrees with the $b_G^*(b_G)$-based equivariant Adams spectral sequence converging to $\pi_*(T(Z))^{\mathbf{Z}/(p)}$ from $E_2$ onward.*

Unfortunately, the Thomified Eilenberg-Moore spectral sequence is known to converge only in the case where the base space in the fibration is simply-connected, from 4.4, which is not the case for the Borel fibration. Note that we would hope that the case of the Thomified Eilenberg-Moore spectral sequence above would converge to $[EG_+, T(Z)]_*^G$, rather than $[S^0, T(Z)]_*^G$, the target of the $\mathbf{Z}/p$-equivariant Adams spectral sequence. Thus, despite the lack of simple-connectivity for the base space, this special case of the Thomified Eilenberg-Moore spectral sequence does converge if $[S^0, T(Z)]_*^G$ is isomorphic to $[EG_+, T(Z)]_*^G = [S^0, F(EG_+, T(Z))]_*^G$ via the comparison map $T(Z) \to F(EG_+, T(Z))$, which is indeed an equivalence when $T(Z)$ is finite, by the (cinfirmed)

Segal Conjecture ([Car84].) Thus, when $Z$ is finite and $G$-free, the Thomified Eilenberg-Moore spectral sequence converges to

$$\pi_*(T(EG_+ \wedge_G Z)) \cong \pi_*(EG_+ \wedge_G T(Z)) \cong \pi_*(T(Z)/G) \cong \pi_*(T(Z)^G),$$

as we wish, where we think of $\pi_*(T(Z))^G$ as $[S^0, T(Z)]^G$ with the sphere $G$-fixed. If $Z$ is not finite, then the Thomified Eilenberg-Moore spectral sequence need not converge. For example, if $T(Z) = EG_+ \wedge H\mathbf{Z}/p$, then $\pi_*(T(Z)) = H_*(G)$, which is bounded below, while $F(EG_+, EG_+ \wedge HG) = F(EG_+, HG)$, which has homotopy $H^*(G)$, which is unbounded. This example was pointed out by the referee.

## References


[Ad84] J. F. Adams. Prerequisites (on equivariant stable homotopy theory) for Carlsson's lecture, *Algebraic Topology, Aarhus 1982,* ed. by I. Madsen and B. Oliver, Lecture Notes Math. 1051:483-532, Springer-Verlag, 1984.

[CE56] H. Cartan and S. Eilenberg. *Homological Algebra.* Princeton University Press, Princeton, 1956.

[Car84] G. Carlsson, Equivariant stable homotopy theory and Segal's Burnside ring conjecture, *Annals of Mathematics* 120:189-224, 1984.

[D74] W. G. Dwyer, Strong convergence of the Eilenberg-Moore spectral sequence, *Topology* 13:255-265, 1974.

[G88] J.P.C. Greenlees, Stable maps into free $G$-spaces, *Trans. A.M.S.* 310:199-215, 1988.

[G90] J.P.C. Greenlees, The power of mod $p$ Borel homology, *Homotopy Theory and Related Topics,* ed. by M. Mimura, Lecture Notes Math. 1418:140-151, Springer-Verlag, 1990.

[LMS86] L.G. Lewis, J.P. May, and M. Steinberger (with contributions by J.E. McClure), *Equivariant Stable Homotopy Theory,* Lecture Notes in Math. 1213, Springer-Verlag 1986.

[Mah79] M. E. Mahowald. Ring spectra which are Thom complexes. *Duke Mathematical Journal*, 46:549–559, 1979.

[MM65] J. W. Milnor and J. C. Moore. On the structure of Hopf algebras. *Annals of Mathematics*, 81(2):211–264, 1965.

[MP67] W. S. Massey and F. P. Peterson. *The* mod *2 cohomology structure of certain fibre spaces*, volume 74 of *Memoirs of the American Mathematical Society*. American Mathematical Society, Providence, Rhode Island, 1967.

[Rav86] D. C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres*. Academic Press, New York, 1986.

[Rec70] D. L. Rector. Steenrod operations in the Eilenberg-Moore spectral sequence. *Commentarii Mathematici Helvetici*, 45:540–552, 1970.

[Smi69] L. Smith. On the construction of the Eilenberg-Moore spectral sequence. *Bulletin of the American Mathematical Society*, 75:873–878, 1969.

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